

INEQUALITIES FOR MODIFIED BESSEL FUNCTIONS AND THEIR INTEGRALS

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ABSTRACT. Simple inequalities for some integrals involving the modified Bessel functions $I_\nu(x)$ and $K_\nu(x)$ are established. We also obtain a monotonicity result for $K_\nu(x)$ and a new lower bound, that involves gamma functions, for $K_0(x)$.

1. INTRODUCTION AND PRELIMINARY RESULTS

In the developing Stein's method for Variance-Gamma distributions, Gaunt [2] required simple bounds, in terms of modified Bessel functions, for the integrals

$$\int_0^x e^{\beta t} t^\nu I_\nu(t) dt \quad \text{and} \quad \int_x^\infty e^{\beta t} t^\nu K_\nu(t) dt,$$

where $x > 0$, $\nu > -1/2$ and $-1 < \beta < 1$. Closed form expressions for these integrals, in terms of modified Bessel functions and the modified Struve function $\mathbf{L}_\nu(x)$, do in fact exist for the case $\beta = 0$. For $z \in \mathbb{C}$ and $\nu \in \mathbb{C}$, let $\mathcal{L}_\nu(z)$ denote $I_\nu(z)$, $e^{\nu\pi i} K_\nu(z)$ or any linear combination of these functions, in which the coefficients are independent of ν and z . From formula 10.43.2 of Olver et. al. [8] we have, for $\nu \neq -1/2$,

$$(1.1) \quad \int z^\nu \mathcal{L}_\nu(z) dz = \sqrt{\pi} 2^{\nu-1} \Gamma(\nu + 1/2) z (\mathcal{L}_\nu(z) \mathbf{L}_{\nu-1}(z) - \mathcal{L}_{\nu-1}(z) \mathbf{L}_\nu(z)).$$

Whilst formula (1.1) holds for complex-valued z and ν , throughout this paper we shall restrict our attention to the case of real-valued z and ν . There are no closed form expressions in terms of modified Bessel and Struve functions in the literature for the integrals for the case $\beta \neq 0$. Moreover, even in the case $\beta = 0$ the expression on the right-hand side of formula (1.1) is a complicated expression involving the modified Struve function $\mathbf{L}_\nu(x)$. This provides the motivation for establishing simple bounds, in terms of modified Bessel functions, for the integrals defined in the first display.

In this paper we establish, through the use of elementary properties of modified Bessel functions and straightforward calculations, simple bounds, that involve modified Bessel functions, for the integrals given in the first display. Our bounds prove to be very useful when applied to calculations that arise in the study of Stein's method for Variance-Gamma distributions. We also obtain a monotonicity result and bound for the modified Bessel function of the second kind $K_\nu(x)$, as well as a simple but remarkably tight lower bound for $K_0(x)$. These bounds are, again,

Date: November 30, 2012 and, in revised form, November 30, 2012.

2000 Mathematics Subject Classification. Primary 33C10.

Key words and phrases. Modified Bessel functions.

The author is supported by an EPSRC DPhil Studentship.

motivated by the need for such bounds in the study of Stein's method for Variance-Gamma distributions. Throughout this paper we make use of some elementary properties of modified Bessel functions and these are stated in the appendix.

2. INEQUALITIES FOR INTEGRALS INVOLVING MODIFIED BESSEL FUNCTIONS

Before presenting our first result concerning inequalities for integrals of modified Bessel functions, we introduce some notation for the repeated integral of the function $e^{\beta x} x^\nu I_\nu(x)$, which will be used in the following theorem. We define

$$(2.1) \quad I_{(\nu, \beta, 0)}(x) = e^{\beta x} x^\nu I_\nu(x), \quad I_{(\nu, \beta, n+1)}(x) = \int_0^x I_{(\nu, \beta, n)}(y) dy, \quad n = 0, 1, 2, 3, \dots$$

With this notation we have:

Theorem 2.1. *Let $0 \leq \gamma < 1$, then the following inequalities hold for all $x > 0$*

$$(2.2) \quad \int_0^x t^\nu I_\nu(t) dt > x^\nu I_{\nu+1}(x), \quad \nu > -1,$$

$$(2.3) \quad \int_0^x t^\nu I_\nu(t) dt < x^\nu I_\nu(x), \quad \nu \geq 1/2,$$

$$(2.4) \quad I_{(\nu, 0, n+1)}(x) < I_{(\nu, 0, n)}(x), \quad \nu \geq 1/2,$$

$$(2.5) \quad I_{(\nu, -\gamma, n)}(x) \leq \frac{1}{(1-\gamma)^n} e^{-\gamma x} I_{(\nu, 0, n)}(x), \quad \nu \geq 1/2, n = 0, 1, 2, \dots,$$

$$(2.6) \quad \int_0^x t^\nu I_{\nu+n}(t) dt < \frac{2(\nu+n+1)}{2\nu+n+1} x^\nu I_{\nu+n+1}(x), \quad \nu > -1/2, n \geq 0,$$

$$(2.7) \quad I_{(\nu, 0, n)}(x) < \left\{ \prod_{k=1}^n \frac{2\nu+2k}{2\nu+k} \right\} x^\nu I_{\nu+n}(x), \quad \nu \geq 0, n = 1, 2, 3, \dots,$$

$$I_{(\nu, -\gamma, n)}(x) < \frac{1}{(1-\gamma)^n} \left\{ \prod_{k=1}^n \frac{2\nu+2k}{2\nu+k} \right\} e^{-\gamma x} x^\nu I_{\nu+n}(x), \quad \nu \geq 1/2, n = 1, 2, 3, \dots$$

Proof. (i) From the differentiation formula (A.12) we have that

$$\int_0^x t^\nu I_\nu(t) dt = \int_0^x \frac{1}{t} t^{\nu+1} I_\nu(t) dt > \frac{1}{x} \int_0^x t^{\nu+1} I_\nu(t) dt = x^\nu I_{\nu+1}(x),$$

since by (A.2) we have $\lim_{x \downarrow 0} x^{\nu+1} I_{\nu+1}(x) = 0$ for $\nu > -1$.

(ii) Using inequality (A.7) and then applying (A.12) we get

$$\int_0^x t^\nu I_\nu(t) dt < \int_0^x t^\nu I_{\nu-1}(t) dt = x^\nu I_\nu(x).$$

(iii) From inequality (2.3), we have

$$I_{(\nu, 0, 1)}(x) < I_{(\nu, 0, 0)}(x).$$

Integrating both sides of the above display n times with respect to x yields the desired inequality.

(iv) We prove the result by induction on n . The result is trivially true for $n = 0$. Suppose the result is true for $n = k$. From the inductive hypothesis we have

$$(2.8) \quad I_{(\nu, -\gamma, k+1)}(x) = \int_0^x I_{(\nu, -\gamma, k)}(t) dt \leq \frac{1}{(1-\gamma)^k} \int_0^x e^{-\gamma t} I_{(\nu, 0, k)}(t) dt.$$

Integration by parts and inequality (2.4) gives

$$\begin{aligned} \int_0^x e^{-\gamma t} I_{(\nu, 0, k)}(t) dt &= e^{-\gamma x} I_{(\nu, 0, k+1)}(x) + \gamma \int_0^x e^{-\gamma t} I_{(\nu, 0, k+1)}(t) dt \\ &< e^{-\gamma x} I_{(\nu, 0, k+1)}(x) + \gamma \int_0^x e^{-\gamma t} I_{(\nu, 0, k)}(t) dt. \end{aligned}$$

Rearranging we obtain

$$\int_0^x e^{-\gamma t} I_{(\nu, 0, k)}(t) dt < \frac{1}{1-\gamma} e^{-\gamma x} I_{(\nu, 0, k+1)}(x),$$

and substituting into (2.8) gives

$$I_{(\nu, -\gamma, k+1)}(x) < \frac{1}{(1-\gamma)^{k+1}} e^{-\gamma x} I_{(\nu, 0, k+1)}(x).$$

Hence the result has been proved by induction.

(v) From the differentiation formula (A.12) and identity (A.10) we get that

$$\begin{aligned} \frac{d}{dt}(t^\nu I_{\nu+n+1}(t)) &= \frac{d}{dt}(t^{-(n+1)} \cdot t^{\nu+n+1} I_{\nu+n+1}(t)) \\ &= t^\nu I_{\nu+n}(t) - (n+1)t^{\nu-1} I_{\nu+n+1}(t) \\ &= t^\nu I_{\nu+n}(t) - \frac{n+1}{2(\nu+n+1)} t^\nu I_{\nu+n}(t) + \frac{n+1}{2(\nu+n+1)} t^\nu I_{\nu+n+2}(t) \\ &= \frac{2\nu+n+1}{2(\nu+n+1)} t^\nu I_{\nu+n}(t) + \frac{n+1}{2(\nu+n+1)} t^\nu I_{\nu+n+2}(t). \end{aligned}$$

Integrating both sides over $(0, x)$, applying the fundamental theorem of calculus and rearranging gives

$$\int_0^x t^\nu I_{\nu+n}(t) dt = \frac{2(\nu+n+1)}{2\nu+n+1} x^\nu I_{\nu+n+1}(x) - \frac{n+1}{2\nu+n+1} \int_0^x t^\nu I_{\nu+n+2}(t) dt.$$

The result now follows from the fact that $I_\nu(x) > 0$ for $x > 0$ and by the positivity of the integral.

(vi) From inequality (2.6) we have

$$I_{(\nu, 0, 1)}(x) = \int_0^x t^\nu I_\nu(t) dt < \frac{2(\nu+1)}{2\nu+1} x^\nu I_{\nu+1}(x),$$

and

$$\begin{aligned} I_{(\nu, 0, 2)}(x) &= \int_0^x I_{(\nu, 0, 1)}(t) dt \\ &< \frac{2(\nu+1)}{2\nu+1} \int_0^x t^\nu I_{\nu+1}(t) dt \\ &< \frac{2(\nu+1)}{2\nu+1} \frac{2(\nu+2)}{2\nu+2} x^\nu I_{\nu+2}(x). \end{aligned}$$

Iterating gives the result.

(vii) This follows from inequalities (2.5) and (2.7). \square

We now establish a simple lemma, which gives a monotonicity result for the ratio $\frac{K_{\nu-1}(x)}{K_\nu(x)}$. The lemma has an immediate corollary, which we will make use of in the proof of our next theorem.

Lemma 2.2. *Suppose $x > 0$, then the function $\frac{K_{\nu-1}(x)}{K_\nu(x)}$ is strictly monotone increasing for $\nu > 1/2$, is constant for $\nu = 1/2$, and is strictly monotone decreasing for $\nu < 1/2$.*

Proof. To simplify the calculations, we let $\mu = \nu + 1/2$ and define $h_\mu(x) = \frac{K_{\mu-1/2}(x)}{K_{\mu+1/2}(x)}$. It follows from the quotient rule and the differentiation formulas (A.15) and (A.16) that

$$(2.9) \quad h'_\mu(x) = -1 + \frac{2\mu}{x}h_\mu(x) + h_\mu^2(x).$$

Now, Theorem 2 of Segura [9] states that

$$(2.10) \quad h'_\mu(x) > \frac{x}{\sqrt{x^2 + \mu^2} + \mu} = \frac{-\mu + \sqrt{x^2 + \mu^2}}{x}, \quad \mu > 0, x > 0,$$

and that for $\mu < 0$ the inequality is reversed and for $\mu = 0$ equality holds. Applying inequality (2.10) to equation (2.9) gives, for $\mu > 0$,

$$h'_\mu(x) > -1 + \frac{2\mu}{x} \left(\frac{-\mu + \sqrt{x^2 + \mu^2}}{x} \right) + \left(\frac{-\mu + \sqrt{x^2 + \mu^2}}{x} \right)^2 = 0$$

Similarly, we see that $h'_0(x) = 0$ and that $h'_\mu(x) < 0$ for $\mu < 0$. This completes the proof. \square

Corollary 2.3. *For $\nu > 1/2$ and $\alpha > 1$ the equation $K_\nu(x) = \alpha K_{\nu-1}(x)$ has one root in the region $x > 0$.*

Proof. From the asymptotic formulas (A.3) and (A.5), it follows that for $\nu > 1/2$,

$$\lim_{x \downarrow 0} \frac{K_{\nu-1}(x)}{K_\nu(x)} = 0, \quad \text{and} \quad \lim_{x \rightarrow \infty} \frac{K_{\nu-1}(x)}{K_\nu(x)} = 1.$$

Since $\frac{K_{\nu-1}(x)}{K_\nu(x)}$ is strictly monotone decreasing on $(0, \infty)$, it follows that for $\alpha > 1$ the equation $K_\nu(x) = \alpha K_{\nu-1}(x)$ (i.e. $\frac{K_{\nu-1}(x)}{K_\nu(x)} = \frac{1}{\alpha}$) has one root in the region $x > 0$. \square

As an aside, we note that Lemma 2.3 allows us to easily establish an inequality for the Turánian $\Delta_\nu(x) = K_\nu^2(x) - K_{\nu-1}(x)K_{\nu+1}(x)$ (for more details on the Turánian $\Delta_\nu(x)$ see Baricz [1]).

Proposition 2.4. *Suppose $x > 0$, then $\Delta_\nu(x) < \Delta_{\nu-1}(x)$ for $\nu > 1/2$, $\Delta_{1/2}(x) = \Delta_{-1/2}(x)$, and $\Delta_\nu(x) > \Delta_{\nu-1}(x)$ for $\nu < 1/2$.*

Proof. By the quotient rule and differentiation formula (A.14), we have

$$\begin{aligned} \frac{d}{dx} \left(\frac{K_{\nu-1}(x)}{K_\nu(x)} \right) &= - \frac{(K_\nu(x) + K_{\nu-2}(x))K_\nu(x) - (K_{\nu+1}(x) + K_{\nu-1}(x))K_{\nu-1}(x)}{2K_\nu^2(x)} \\ &= \frac{K_{\nu-1}^2(x) - K_{\nu-2}(x)K_\nu(x) - (K_\nu^2(x) - K_{\nu-1}(x)K_{\nu+1}(x))}{2K_\nu^2(x)} \\ &= \frac{\Delta_{\nu-1}(x) - \Delta_\nu(x)}{2K_\nu^2(x)}. \end{aligned}$$

Since, by Lemma 2.3, the function $\frac{K_{\nu-1}(x)}{K_{\nu}(x)}$ is strictly monotone increasing for $\nu > 1/2$, is constant for $\nu = 1/2$, and is strictly monotone decreasing for $\nu < 1/2$, the result follows. \square

With the aid of Corollary 2.3 and standard properties of the modified Bessel function $K_{\nu}(x)$, we can prove at the following theorem.

Theorem 2.5. *Let $-1 < \beta < 1$, then for all $x > 0$ the following inequalities hold*

$$(2.11) \quad \begin{aligned} \int_x^{\infty} t^{\nu} K_{\nu}(t) dt &< x^{\nu} K_{\nu+1}(x), \quad \nu \in \mathbb{R}, \\ \int_x^{\infty} t^{\nu} K_{\nu}(t) dt &< x^{\nu} K_{\nu}(x), \quad \nu < 1/2, \end{aligned}$$

$$(2.12) \quad \int_x^{\infty} e^{\beta t} t^{\nu} K_{\nu}(t) dt < \frac{1}{1-|\beta|} e^{\beta x} x^{\nu} K_{\nu}(x), \quad \nu < 1/2,$$

$$(2.13) \quad \begin{aligned} \int_x^{\infty} t^{\nu} K_{\nu}(t) dt &\leq \frac{\sqrt{\pi} \Gamma(\nu + 1/2)}{\Gamma(\nu)} x^{\nu} K_{\nu}(x), \quad \nu \geq 1/2, \\ \int_x^{\infty} e^{\beta t} t^{\nu} K_{\nu}(t) dt &\leq \frac{2\sqrt{\pi} \Gamma(\nu + 1/2)}{(1-\beta^2)^{\nu+1/2} \Gamma(\nu)} e^{\beta x} x^{\nu} K_{\nu}(x), \quad \nu \geq 1/2. \end{aligned}$$

Proof. (i) From the differentiation formula (A.13) we have that

$$\int_x^{\infty} t^{\nu} K_{\nu}(t) dt = \int_x^{\infty} \frac{1}{t} t^{\nu+1} K_{\nu}(t) dt < \frac{1}{x} \int_x^{\infty} t^{\nu+1} K_{\nu}(t) dt = x^{\nu} K_{\nu+1}(x),$$

since, by the asymptotic formula (A.5), $\lim_{x \rightarrow \infty} x^{\nu+1} K_{\nu+1}(x) = 0$.

(ii) Using inequality (A.8) and then apply the differentiation formula (A.13) we have

$$\int_x^{\infty} t^{\nu} K_{\nu}(t) dt < \int_x^{\infty} t^{\nu} K_{\nu-1}(t) dt = x^{\nu} K_{\nu}(x).$$

(iii) Now suppose that $\nu < 1/2$ and $\beta > 0$. Using integration by parts and the differentiation formula (A.13) gives

$$(2.14) \quad \int_x^{\infty} e^{\beta t} t^{\nu} K_{\nu}(t) dt = -\frac{1}{\beta} e^{\beta x} x^{\nu} K_{\nu}(x) + \frac{1}{\beta} \int_x^{\infty} e^{\beta t} t^{\nu} K_{\nu-1}(t) dt.$$

Applying the inequality (A.8) and rearranging gives

$$\left(\frac{1}{\beta} - 1\right) \int_x^{\infty} e^{\beta t} t^{\nu} K_{\nu}(t) dt < \frac{1}{\beta} e^{\beta x} x^{\nu} K_{\nu}(x).$$

Inequality (2.12) for $\beta > 0$ now follows on rearranging.

The case $\beta \leq 0$ is simple. Since $e^{\beta t}$ is a non increasing function of t when $\beta \leq 0$ we have

$$\int_x^{\infty} e^{\beta t} t^{\nu} K_{\nu}(t) dt \leq e^{\beta x} \int_x^{\infty} t^{\nu} K_{\nu}(t) dt < e^{\beta x} x^{\nu} K_{\nu}(x) \leq \frac{1}{1-|\beta|} e^{\beta x} x^{\nu} K_{\nu}(x),$$

where we used inequality (2.11) to obtain the the second inequality. Hence inequality (2.12) has been proved.

(iv) The case $\nu = 1/2$ is simple. Using (A.1) we may easily integrate $t^{1/2} K_{1/2}(t)$:

$$\int_x^{\infty} t^{1/2} K_{1/2}(t) dt = \int_x^{\infty} \sqrt{\frac{\pi}{2}} e^{-t} dt = \sqrt{\frac{\pi}{2}} e^{-x} = x^{1/2} K_{1/2}(x).$$

It therefore follows that inequality (2.13) holds for $\nu = 1/2$ because we have

$$\frac{\sqrt{\pi}\Gamma(1)}{\Gamma(1/2)} = 1,$$

where we used the facts that $\Gamma(1) = 1$ and $\Gamma(1/2) = \sqrt{\pi}$.

Now suppose $\nu > 1/2$. We begin by defining the function $u(x)$ to be

$$u(x) = Mx^\nu K_\nu(x) - \int_x^\infty t^\nu K_\nu(t) dt,$$

where

$$M = \frac{\sqrt{\pi}\Gamma(\nu + 1/2)}{\Gamma(\nu)}.$$

We now show that $u(x) \geq 0$ for all $x \geq 0$, which will prove the result. We begin by noting that $\lim_{x \rightarrow 0^+} u(x) = 0$ and $\lim_{x \rightarrow \infty} u(x) = 0$, which are verified by the following calculations, where we make use of the asymptotic formula (A.3) and the definite integral formula (A.17).

$$\begin{aligned} u(0) &= \lim_{x \rightarrow 0^+} \frac{\sqrt{\pi}\Gamma(\nu + 1/2)}{\Gamma(\nu)} x^\nu K_\nu(x) - \int_0^\infty t^\nu K_\nu(t) dt \\ &= \sqrt{\pi}\Gamma(\nu + 1/2)2^{\nu-1} - \sqrt{\pi}\Gamma(\nu + 1/2)2^{\nu-1} \\ &= 0, \end{aligned}$$

and

$$\lim_{x \rightarrow \infty} u(x) = \lim_{x \rightarrow \infty} Mx^\nu K_\nu(x) - \lim_{x \rightarrow \infty} \int_x^\infty t^\nu K_\nu(t) dt = 0,$$

where we used the asymptotic formula (A.5) to obtain the above equality. We may obtain an expression for the first derivative of $u(x)$ by the use of the differentiation formula (A.13) as follows

$$(2.15) \quad u'(x) = x^\nu [K_\nu(x) - MK_{\nu-1}(x)].$$

In the limit $x \rightarrow 0^+$ we have, by the asymptotic formula (A.3), that

$$u'(x) \sim \begin{cases} x^\nu \left\{ 2^{\nu-1}\Gamma(\nu) \frac{1}{x^\nu} - M2^{|\nu-1|-1}\Gamma(|\nu-1|) \frac{1}{x^{|\nu-1|}} \right\}, & \nu \neq 1, \\ x^\nu \left\{ 2^{\nu-1}\Gamma(\nu) \frac{1}{x^\nu} + M \log x \right\}, & \nu = 1. \end{cases}$$

Since $\nu > |\nu - 1|$ for $\nu > 1/2$ and $\lim_{x \rightarrow 0^+} x^a \log x = 0$, where $a > 0$, we have

$$u'(x) \sim 2^{\nu-1}\Gamma(\nu), \quad \text{as } x \rightarrow 0^+, \quad \text{for } \nu > 1/2.$$

Therefore $u(x)$ is initially an increasing function of x . In the limit $x \rightarrow \infty$ we have, by (A.5),

$$u'(x) \sim \left(1 - \frac{\sqrt{\pi}\Gamma(\nu + 1/2)}{\Gamma(\nu)} \right) \sqrt{\frac{\pi}{2}} x^{\nu-1/2} e^{-x} < 0, \quad \text{for } \nu > 1/2.$$

We therefore see that $u(x)$ is an decreasing function of x for large, positive x . From the formula (2.15) we see that x^* is a turning point of $u(x)$ if and only if

$$(2.16) \quad K_\nu(x^*) = \frac{\sqrt{\pi}\Gamma(\nu + 1/2)}{\Gamma(\nu)} K_{\nu-1}(x^*).$$

From Corollary 2.3, it follows that equation (2.16) has one root for $\nu > 1/2$ (for which $\frac{\sqrt{\pi}\Gamma(\nu+1/2)}{\Gamma(\nu)} > 1$).

Putting these results together, we see that $u(x)$ is non-negative at the origin and initially increases until it reaches its maximum value at x^* , it then decreases and tends to 0 as $x \rightarrow \infty$. Therefore $u(x)$ is non-negative for all $x \geq 0$ when $\nu > 1/2$.

(v) The proof for $\beta \leq 0$ is easy and follows immediately from part (iv), since $1 < \frac{2}{(1-\beta^2)^{\nu+1/2}}$ for $\nu \geq 1/2$. So we suppose $\beta > 0$. Again, because $K_{1/2}(x) = \sqrt{\frac{\pi}{2x}}e^{-x}$, the case $\nu = 1/2$ is straightforward, so we also suppose $\nu > 1/2$. We make use of a similar argument to the one used in the proof of part (iv). We define the function $v(x)$ to be

$$v(x) = Ne^{\beta x}x^\nu K_\nu(x) - \int_x^\infty e^{\beta t}t^\nu K_\nu(t)dt,$$

where

$$N = \frac{2\sqrt{\pi}\Gamma(\nu+1/2)}{(1-\beta^2)^{\nu+1/2}\Gamma(\nu)}.$$

We now show that $v(x) \geq 0$ for all $x \geq 0$, which will prove the result. We begin by noting that $\lim_{x \rightarrow 0^+} v(x) > 0$ and $\lim_{x \rightarrow \infty} v(x) = 0$, which are verified by the following calculations, where we make use of the asymptotic formula (A.3) and the definite integral formula (A.17).

$$\begin{aligned} v(0) &= \lim_{x \rightarrow 0^+} \frac{2\sqrt{\pi}\Gamma(\nu+1/2)}{(1-\beta^2)^{\nu+1/2}\Gamma(\nu)}x^\nu K_\nu(x) - \int_0^\infty e^{\beta t}t^\nu K_\nu(t)dt \\ &= \frac{2\sqrt{\pi}\Gamma(\nu+1/2)}{(1-\beta^2)^{\nu+1/2}\Gamma(\nu)} \cdot 2^{\nu-1}\Gamma(\nu) - \int_0^\infty e^{\beta t}t^\nu K_\nu(t)dt \\ &> \frac{2\sqrt{\pi}\Gamma(\nu+1/2)}{(1-\beta^2)^{\nu+1/2}\Gamma(\nu)} \cdot 2^{\nu-1}\Gamma(\nu) - \int_{-\infty}^\infty e^{\beta t}|t|^\nu K_\nu(|t|)dt \\ &= \frac{\sqrt{\pi}\Gamma(\nu+1/2)2^\nu}{(1-\beta^2)^{\nu+1/2}} - \frac{\sqrt{\pi}\Gamma(\nu+1/2)2^\nu}{(1-\beta^2)^{\nu+1/2}} \\ &= 0, \end{aligned}$$

and

$$\lim_{x \rightarrow \infty} v(x) = \lim_{x \rightarrow \infty} Ne^{\beta x}x^\nu K_\nu(x) - \lim_{x \rightarrow \infty} \int_x^\infty e^{\beta t}t^\nu K_\nu(t)dt = 0,$$

where we used the asymptotic formula (A.5) to obtain the above equality. We may obtain an expression for the first derivative of $v(x)$ by the use of the differentiation formula (A.13) as follows

$$(2.17) \quad v'(x) = e^{\beta x}x^\nu[(1+N\beta)K_\nu(x) - NK_{\nu-1}(x)].$$

In the limit $x \rightarrow 0^+$ we have, by the asymptotic formula (A.3), that

$$v'(x) \sim \begin{cases} e^{\beta x}x^\nu \left\{ 2^{\nu-1}\Gamma(\nu)(1+N\beta)\frac{1}{x^\nu} - N \cdot 2^{|\nu-1|-1}\Gamma(|\nu-1|)\frac{1}{x^{|\nu-1|}} \right\}, & \nu \neq 1, \\ e^{\beta x}x^\nu \left\{ 2^{\nu-1}\Gamma(\nu)(1+N\beta)\frac{1}{x^\nu} + N \log x \right\}, & \nu = 1. \end{cases}$$

As in part (iv), we see that $v(x)$ is initially an increasing function of x . In the limit $x \rightarrow \infty$ we have

$$v'(x) \sim (1-N(1-\beta))\sqrt{\frac{\pi}{2}}x^{\nu-1/2}e^{(\beta-1)x}, \quad \text{for } \nu > 1/2.$$

Now, for $\nu > 1/2$ and $0 < \beta < 1$ we have, by (A.5),

$$(2.18) \quad N(1-\beta) = \frac{2\sqrt{\pi}\Gamma(\nu+1/2)}{\Gamma(\nu)} \cdot \frac{1}{(1-\beta^2)^{\nu-1/2}} \cdot \frac{1}{1+\beta} > 2 \cdot 1 \cdot \frac{1}{2} = 1.$$

Hence, $v(x)$ is an decreasing function of x for large, positive x . From formula (2.17) we see that x^* is a turning point of $v(x)$ if and only if

$$(2.19) \quad (1 + N\beta)K_\nu(x^*) = NK_{\nu-1}(x^*).$$

Inequality (2.18) shows that $N > 1 + N\beta$ for all $\nu > 1/2$ and $0 < \beta < 1$. From Corollary 2.3, it follows that equation (2.19) has one root for positive x and therefore $v(x)$ has one maximum which occurs at positive x . Putting these results together we see that $v(x)$ is positive at the origin and initially increases until it reaches its maximum value at x^* , it then decreases and tends to 0 as $x \rightarrow \infty$. Therefore $v(x)$ is non-negative for all $x \geq 0$ when $\nu > 1/2$, which completes the proof. \square

Combining the inequalities of Theorems 2.1 and 2.5 and the indefinite integral formula (1.1) we may obtain lower and upper bounds for the quantity $\mathcal{L}_\nu(x)\mathbf{L}_{\nu-1}(x) - \mathcal{L}_{\nu-1}(x)\mathbf{L}_\nu(x)$. Here is an example:

Corollary 2.6. *Suppose $\nu > -1/2$, then for all $x > 0$ we have*

$$\frac{x^{\nu-1}I_{\nu+1}(x)}{\sqrt{\pi}2^{\nu-1}\Gamma(\nu+1/2)} < I_\nu(x)\mathbf{L}_{\nu-1}(x) - I_{\nu-1}(x)\mathbf{L}_\nu(x) < \frac{(\nu+1)x^{\nu-1}I_{\nu+1}(x)}{\sqrt{\pi}2^{\nu-1}\Gamma(\nu+3/2)}.$$

Proof. From the asymptotic formulas (A.2) and (A.6) for $I_\nu(x)$ and $\mathbf{L}(x)$, respectively, we have that

$$\lim_{x \downarrow 0} (x(I_\nu(x)\mathbf{L}_{\nu-1}(x) - I_{\nu-1}(x)\mathbf{L}_\nu(x))) = 0, \quad \text{for } \nu > -1/2.$$

Therefore, applying the indefinite integral formula (1.1) gives, for $\nu > -1/2$,

$$(2.20) \quad \int_0^x t^\nu I_\nu(t) dt = \sqrt{\pi}2^{\nu-1}\Gamma(\nu+1/2)x(I_\nu(x)\mathbf{L}_{\nu-1}(x) - I_{\nu-1}(x)\mathbf{L}_\nu(x)).$$

From inequalities (2.2) and (2.6) of Theorem 2.1, we have

$$x^\nu I_{\nu+1}(x) < \int_0^x t^\nu I_\nu(t) dt < \frac{2(\nu+1)}{2\nu+1}x^\nu I_{\nu+1}(x).$$

Substituting this inequality into (2.20) gives

$$\begin{aligned} x^\nu I_{\nu+1}(x) &< \sqrt{\pi}2^{\nu-1}\Gamma(\nu+1/2)x(I_\nu(x)\mathbf{L}_{\nu-1}(x) - I_{\nu-1}(x)\mathbf{L}_\nu(x)) \\ &< \frac{2(\nu+1)}{2\nu+1}x^\nu I_{\nu+1}(x). \end{aligned}$$

The desired inequality now follows from rearranging terms and an application of the standard formula $x\Gamma(x) = \Gamma(x+1)$. \square

Remark 2.7. The lower and upper bounds for $I_\nu(x)\mathbf{L}_{\nu-1}(x) - I_{\nu-1}(x)\mathbf{L}_\nu(x)$ that are given in Corollary 2.6 are simple, but very tight for large ν .

3. INEQUALITIES FOR THE MODIFIED BESSEL FUNCTION OF THE SECOND KIND

We now present some simple inequalities for the modified Bessel function of the second kind $K_\nu(x)$. The following theorem establishes an inequality for the modified Bessel function $K_\nu(x)$ that is useful in the study of Stein's method for Variance-Gamma distributions (see Gaunt [2]).

Theorem 3.1. *Let $\mu > 1$ and $x \geq 0$, then*

$$(3.1) \quad \frac{1}{x^2} - \frac{x^{\mu-2}K_\mu(x)}{2^{\mu-1}\Gamma(\mu)},$$

is a monotone decreasing function of x on $(0, \infty)$ and satisfies the following inequality

$$(3.2) \quad 0 < \frac{1}{x^2} - \frac{x^{\mu-2}K_\mu(x)}{2^{\mu-1}\Gamma(\mu)} \leq \frac{1}{4(\mu-1)}, \quad \text{for all } x \geq 0.$$

Proof. Applying the differentiation formula (A.14) gives

$$(3.3) \quad \begin{aligned} & \frac{d}{dx} \left(\frac{1}{x^2} - \frac{x^{\mu-2}K_\mu(x)}{2^{\mu-1}\Gamma(\mu)} \right) \\ &= -\frac{2}{x^3} - \frac{(\mu-2)x^{\mu-3}K_\mu(x) - \frac{1}{2}(K_{\mu-1}(x) + K_{\mu+1}(x))x^{\mu-2}}{2^{\mu-1}\Gamma(\mu)}. \end{aligned}$$

Using (A.11) we may simplify the numerator as follows

$$\begin{aligned} & (\mu-2)K_\mu(x) - \frac{1}{2}x(K_{\mu-1}(x) + K_{\mu+1}(x)) \\ &= (\mu-2)K_\mu(x) - \frac{1}{2}x \left(2K_{\mu-1}(x) + \frac{2\mu}{x}K_\mu(x) \right) \\ &= -xK_{\mu-1}(x) - 2K_\mu(x). \end{aligned}$$

Hence, (3.3) simplifies to

$$\frac{d}{dx} \left(\frac{1}{x^2} - \frac{x^{\mu-2}K_\mu(x)}{2^{\mu-1}\Gamma(\mu)} \right) = \frac{-2^\mu\Gamma(\mu) + x^{\mu+1}K_{\mu-1}(x) + 2x^\mu K_\mu(x)}{2^{\mu-1}\Gamma(\mu)x^3}.$$

Thus, proving that (3.1) is monotone decreasing reduces to proving that, for $x > 0$,

$$(3.4) \quad x^{\mu+1}K_{\mu-1}(x) + 2x^\mu K_\mu(x) < 2^\mu\Gamma(\mu).$$

From (A.13) we get that

$$\begin{aligned} \frac{d}{dx} (x^{\mu+1}K_{\mu-1}(x) + 2x^\mu K_\mu(x)) &= \frac{d}{dx} (x^2 \cdot x^{\mu-1}K_{\mu-1}(x) + 2x^\mu K_\mu(x)) \\ &= 2x^\mu K_{\mu-1}(x) - x^{\mu+1}K_{\mu-2}(x) - 2x^\mu K_{\mu-1}(x) \\ &= -x^{\mu+1}K_{\mu-2}(x) \\ &< 0. \end{aligned}$$

So $x^{\mu+1}K_{\mu-1}(x) + 2x^\mu K_\mu(x)$ is a monotone decreasing function of x and from the asymptotic formula (A.3) we see that its limit as $x \rightarrow 0^+$ is $\lim_{x \rightarrow 0^+} (x^{\mu+1}K_{\mu-1}(x) + 2x^\mu K_\mu(x)) = 2 \cdot 2^{\mu-1}\Gamma(\mu) = 2^\mu\Gamma(\mu)$. Therefore (3.4) is proved, and so (3.1) is monotone decreasing on $(0, \infty)$. It is therefore bounded above and below its values in the limits $x \rightarrow \infty$ and $x \rightarrow 0$. These are calculated using the asymptotic formulas (A.5) and (A.4) and are given below:

$$\begin{aligned} \lim_{x \rightarrow \infty} \left(\frac{1}{x^2} - \frac{x^{\mu-2}K_\mu(x)}{2^{\mu-1}\Gamma(\mu)} \right) &= 0, \\ \lim_{x \rightarrow 0^+} \left(\frac{1}{x^2} - \frac{x^{\mu-2}K_\mu(x)}{2^{\mu-1}\Gamma(\mu)} \right) &= \frac{2^{\mu-3}\Gamma(\mu-1)}{2^{\mu-1}\Gamma(\mu)} = \frac{1}{4(\mu-1)}. \end{aligned}$$

This completes the proof. \square

Finally, we establish a simple, but surprisingly tight, lower bound for the modified Bessel function $K_0(x)$.

Theorem 3.2. *Let $x > 0$, then*

$$(3.5) \quad \frac{\Gamma(x+1/2)}{\Gamma(x+1)} < \sqrt{\frac{2}{\pi}} e^x K_0(x).$$

Proof. Formula 10.32.8 of Olver et. al. [8] gives the following integral representation of $K_0(x)$:

$$K_0(x) = \int_1^\infty \frac{e^{-xt}}{\sqrt{t^2-1}} dt, \quad x > 0.$$

Setting $t = 2u + 1$ gives

$$K_0(x) = e^{-x} \int_0^\infty \frac{e^{-2xu}}{\sqrt{u^2+u}} du.$$

For $u > 0$ we have $e^{2u} - 1 = \sum_{k=1}^\infty \frac{(2u)^k}{k!} > 2u + 2u^2$, and so

$$e^x K_0(x) > \sqrt{2} \int_0^\infty \frac{e^{-2xu}}{\sqrt{e^{2u}-1}} du = \sqrt{2} \int_0^\infty \frac{e^{-(2x+1)u}}{\sqrt{1-e^{-2u}}} du, \quad \text{for } x > 0.$$

Making the the change of variables $y = e^{-2u}$ gives

$$\begin{aligned} \sqrt{2} \int_0^\infty \frac{e^{-(2x+1)u}}{\sqrt{1-e^{-2u}}} du &= \frac{1}{\sqrt{2}} \int_0^1 (1-y)^{-1/2} y^{x-1/2} dy \\ &= \frac{1}{\sqrt{2}} B(1/2, x+1/2) \\ &= \frac{\Gamma(1/2)\Gamma(x+1/2)}{\sqrt{2}\Gamma(x+1)} \\ &= \frac{\sqrt{\pi}\Gamma(x+1/2)}{\sqrt{2}\Gamma(x+1)}, \end{aligned}$$

where $B(a, b)$ is the beta function, and we used the standard formula $B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ to obtain the third equality. This completes the proof. \square

Corollary 3.3. *Let $x > 0$, then*

$$\frac{1}{\sqrt{x+1/2}} < \sqrt{\frac{2}{\pi}} e^x K_0(x) < \frac{1}{\sqrt{x}}.$$

Proof. The upper bound follows because $K_0(x) < K_{1/2}(x) = \sqrt{\frac{\pi}{2x}} e^{-x}$. The lower bound follows since $\frac{\Gamma(x+1/2)}{\Gamma(x+1)} > \frac{1}{\sqrt{x+1/2}}$, which we now prove. Examining the proof of Theorem 3.2 we see that

$$\frac{\Gamma(x+1/2)}{\Gamma(x+1)} = \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{e^{-(2x+1)u}}{\sqrt{1-e^{-2u}}} du.$$

Now, for $u > 0$ we have $1 - e^{-2u} = \sum_{k=1}^\infty (-1)^{k+1} \frac{(2u)^k}{k!} < 2u$, and so

$$\frac{\Gamma(x+1/2)}{\Gamma(x+1)} > \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{e^{-(2x+1)u}}{\sqrt{2u}} du = \frac{2\sqrt{2}}{\sqrt{\pi}} \int_0^\infty e^{-(2x+1)v^2} dv = \frac{1}{\sqrt{x+1/2}},$$

as required \square

Remark 3.4. Luke [6] obtained the following bounds for $K_0(x)$:

$$\frac{8\sqrt{x}}{8x+1} < \sqrt{\frac{2}{\pi}} e^x K_0(x) < \frac{16x+7}{(16x+9)\sqrt{x}}.$$

Numerical experiments show that the bounds of Luke and our lower bound of Corollary 3.2 are remarkably accurate for all but very small x , for which the logarithmic singularity of $K_0(x)$ blows up. The lower bound $\frac{8\sqrt{x}}{8x+1}$ outperforms our bound lower bound of $\frac{\Gamma(x+1/2)}{\Gamma(x+1)}$ for $x > 0.394$ (3 d.p.), whilst our bound outperforms for $x < 0.394$ (3 d.p.), and performs considerably better for very small x .

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APPENDIX A. ELEMENTARY OF PROPERTIES MODIFIED BESSEL FUNCTIONS

Here we list standard properties of modified Bessel functions that are used throughout this paper. All these formulas can be found in Olver et al. [8], except for the inequalities and the integration formula (A.17), which can be found in Gradshteyn and Ryzhik [3].

A.1. Basic properties. The modified Bessel functions $I_\nu(x)$ and $K_\nu(x)$ are both regular functions of x . They satisfy the following simple inequalities

$$I_\nu(x) > 0 \quad \text{for all } x > 0, \text{ for } \nu > -1,$$

$$K_\nu(x) > 0 \quad \text{for all } x > 0, \text{ for all } \nu \in \mathbb{R}.$$

A.2. Spherical Bessel functions.

$$(A.1) \quad K_{1/2}(x) = K_{-1/2}(x) = \sqrt{\frac{\pi}{2x}} e^{-x}.$$

A.3. Asymptotic expansions.

$$(A.2) \quad I_\nu(x) \sim \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^\nu, \quad x \downarrow 0, \nu > -1,$$

$$(A.3) \quad K_\nu(x) \sim \begin{cases} 2^{|\nu|-1} \Gamma(|\nu|) x^{-|\nu|}, & x \downarrow 0, \nu \neq 0, \\ -\log x, & x \downarrow 0, \nu = 0, \end{cases}$$

$$(A.4) \quad K_\nu(x) \sim 2^{\nu-1} \Gamma(\nu) x^{-\nu} - 2^{\nu-3} \Gamma(\nu-1) x^{-\nu+2}, \quad x \downarrow 0, \nu > 1,$$

$$(A.5) \quad K_\nu(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x}, \quad x \rightarrow \infty,$$

$$(A.6) \quad \mathbf{L}_\nu(x) \sim \frac{2}{\sqrt{\pi} \Gamma(\nu+3/2)} \left(\frac{x}{2}\right)^{\nu+1}, \quad x \downarrow 0, \nu > -1/2.$$

A.4. Inequalities. Let $x > 0$ then following inequalities hold

$$(A.7) \quad I_\nu(x) < I_{\nu-1}(x), \quad \nu \geq 1/2,$$

$$(A.8) \quad K_\nu(x) < K_{\nu-1}(x), \quad \nu < 1/2,$$

$$(A.9) \quad K_\nu(x) \geq K_{\nu-1}(x), \quad \nu \geq 1/2.$$

We have equality in (A.9) if and only if $\nu = 1/2$. The inequalities for $K_\nu(x)$ can be found in Ifantis and Siafarikas [4], whilst the inequality for $I_\nu(x)$ can be found in Jones [5] and Näsell [7].

A.5. Identities.

$$(A.10) \quad I_{\nu+1}(x) = I_{\nu-1}(x) - \frac{2\nu}{x} I_\nu(x),$$

$$(A.11) \quad K_{\nu+1}(x) = K_{\nu-1}(x) + \frac{2\nu}{x} K_\nu(x).$$

A.6. Differentiation.

$$(A.12) \quad \frac{d}{dx}(x^\nu I_\nu(x)) = x^\nu I_{\nu-1}(x),$$

$$(A.13) \quad \frac{d}{dx}(x^\nu K_\nu(x)) = -x^\nu K_{\nu-1}(x),$$

$$(A.14) \quad \frac{d}{dx}(K_\nu(x)) = -\frac{1}{2}(K_{\nu+1}(x) + K_{\nu-1}(x)),$$

$$(A.15) \quad \frac{d}{dx}(K_\nu(x)) = -K_{\nu-1}(x) - \frac{\nu}{x} K_\nu(x),$$

$$(A.16) \quad \frac{d}{dx}(K_\nu(x)) = -K_{\nu+1}(x) + \frac{\nu}{x} K_\nu(x).$$

A.7. Integration.

$$(A.17) \quad \int_{-\infty}^{\infty} e^{\beta t} |t|^\nu K_\nu(|t|) dt = \frac{\sqrt{\pi} \Gamma(\nu+1/2) 2^\nu}{(1-\beta^2)^{\nu+1/2}}, \quad \nu > -1/2, -1 < \beta < 1.$$